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Singularities of the discriminant of pairs of regular foliations in \mathbb{R}^3

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1 Introduction

An important feature of pairs of codimension one regular foliations in \mathbb{R}^3 is its discriminant which is the locus of points where the foliations are tangent. Assuming that the foliations are the leaves of germs of differential 1-forms ω and η , then the discriminant $D(\omega, \eta)$ of the pair (ω, η) is the zero locus of $\omega \wedge \eta$, that is, the locus where the 1-form ω is a multiple of η . This is generically a germ of a space curve. In local coordinates, the discriminant is given by the fibre of a map-germ (the discriminant map-germ) $F : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0)$. In [2] the authors show that the discriminant $D(\omega, \eta)$ determines the local topological type of the pair (ω, η) and obtain a complete list of discrete topological models.

More precisely, assume that the discriminant $D(\omega, \eta)$ is transverse away from 0 to the pair of foliations. When the number of branches of the discriminant in each half-region delimited by the leaf of ω (or η) through 0 is at most 2, then it is showed in [2] that the local topological type is determined by their configuration. The authors also remark that if the number of the branches of the discriminant in a half-region delimited by the leaf of ω (or η) through 0 is more than 2, then there is no discrete topological model.

Since the discriminant plays a key role in the topological classification of the pair (ω, η) , it is of interest to analyse its singularities. This is carried out in [2, 3]. We give in this paper a résumé on the various classifications that are given in those articles.

2 Classifications

In the coordinates system (x, y, z) , we can set $\omega = df$ and $\eta = dz$, where $f : \mathbb{R}^3, 0 \rightarrow \mathbb{R}, 0$ is a germ of a smooth function. Then $D(\omega, \eta)$ is the zero fibre of the map-germ $F_{\omega, \eta} : \mathbb{R}^3, 0 \rightarrow \mathbb{R}^2$ given by $F_{\omega, \eta}(x, y, z) = (f_x(x, y, z), f_y(x, y, z))$.

Let \mathcal{E}_n be the local ring of germs of functions $\mathbb{R}^n, 0 \rightarrow \mathbb{R}$ and m_n its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n, p)$ the p -tuples

of elements in \mathcal{E}_n . Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = \text{Diff}(\mathbb{R}^n, 0) \times \text{Diff}(\mathbb{R}^p, 0)$ denotes the group of right-left equivalence which acts smoothly on $m_n\mathcal{E}(n, p)$ by $(h, k).G = k \circ G \circ h^{-1}$. We have another group of interest, namely the contact group \mathcal{K} . The group \mathcal{K} is the set of germs of diffeomorphisms $H = (h, H_1) \in \text{Diff}(\mathbb{R}^{n+p}, 0)$, with $h \in \text{Diff}(\mathbb{R}^n, 0)$. Then H acts on $m_n\mathcal{E}(n, p)$ as follows: $G = H.F$ if and only if $H(x, F(x)) = (h(x), G(h(x)))$.

2.1 \mathcal{K} -singularities of the discriminant

The action of the group \mathcal{K} is a natural one to use when one seeks to understand the singularities of the zero fibres of germs in $m_n\mathcal{E}(n, p)$.

When at least one foliation is regular, the description of the singularities of the discriminant of the pair (ω, η) via an action of the contact group \mathcal{K} on $m_3\mathcal{E}(3, 2)$ reduces to an action on families of matrices. The corank 2 simple germs are given in Table 1 ([1]). See [2], Sect. 5.1 for details.

Table 1: The \mathcal{K} -simples singularities of corank 2 germs $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ ([1]).

| Name | Normal form | \mathcal{K}_e - codimension |
|-----------|-----------------------------------|-------------------------------|
| $P_{2,k}$ | $(xy, x^k + y^2 + z^2), k \geq 2$ | $k + 1$ |
| $P_{3,3}$ | $(xy, x^3 + y^3 + z^2)$ | 5 |
| $P_{3,4}$ | $(xy, x^4 + y^3 + z^2)$ | 6 |
| $P_{3,5}$ | $(xy, x^5 + y^3 + z^2)$ | 7 |
| J_7 | $(y^2 + xz, x^3 + yz)$ | 5 |
| J_8 | $(y^2 + xz, x^2y + yz)$ | 6 |
| J_9 | $(y^2 + xz, x^4 + yz)$ | 7 |
| K_8 | $(y^2 + xz, x^3 + z^2)$ | 6 |
| K_9 | $(y^2 + xz, x^2y + z^2)$ | 7 |
| G_{10} | $(y^2 + x^3, x^3 + z^2)$ | 7 |
| G_{11} | $(y^2 + x^3, x^2y + z^2)$ | 8 |

2.2 \mathcal{K}_V -singularities of the discriminant

The action of the subgroup \mathcal{K}_V of \mathcal{K} on $m_3\mathcal{E}(3, 2)$, where the changes of coordinates in the source preserve the variety V given by $z = 0$, preserves the smooth type of the zero fibre of $F : \mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ as well as the leaf $z = 0$ of dz . Since the key ingredient in the topological classification of the pairs (df, dz) is the number of branches of the discriminant in each semi-space delimited by the plane $z = 0$, it is natural to seek a classification of F under such action of \mathcal{K}_V .

Theorem 2.1 (Theorem 5.1, [2]) *The \mathcal{K}_V -simple map-germ $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ are given in Table 2.*

It follows that the \mathcal{K}_V -simple singularities of the discriminant $D(df, dz)$ of pairs (df, dz) are given in Table 2. The action of the subgroup \mathcal{K}_V of \mathcal{K} preserves the number of branches, in each semi-space $z > 0$ and $z < 0$, of the zero fibre of any map-germ $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$.

Table 2: Normal forms of \mathcal{K}_V -simple map-germs $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$.

| Normal form F | $d_e(F, \mathcal{K}_V)$ |
|-------------------------------|-------------------------|
| (x, y) | 0 |
| $(x, z + y^k), k \geq 2$ | $k - 2$ |
| $(x, y^2 + z^k), k \geq 2$ | $k - 1$ |
| $(x, y^2 - z^{2k}), k \geq 1$ | $2k - 1$ |
| $(x, yz + y^k), k \geq 3$ | $k - 1$ |
| $(x, z^2 + y^3)$ | 3 |

2.3 \mathcal{K}^* -singularities of the discriminant

It is of interest to classify the singularities of the discriminant up to an equivalence that preserves the leaves of the foliation ω or η . See [2], Sect. 5.3. Let \mathcal{K}^* be the subgroup of \mathcal{K} (and indeed of \mathcal{K}_V) where the changes of coordinates in the source preserve the foliation defined by dz , i.e., the horizontal planes. Recall that the discriminant is the zero fibre of map germ $F_{\omega, \eta} : \mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$, so we are seeking a classification of the singularities of map-germs $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ under the action of the subgroup \mathcal{K}^* , which is a Damon geometric subgroup. The classification of the \mathcal{K}^* -simple map-germs $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ coincide with the \mathcal{K}_V -simple singularities.

Theorem 2.2 (Theorem 5.4, [2]) *A map-germ $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$ is \mathcal{K}^* -simple if and only if it is \mathcal{K}_V -simple. So the \mathcal{K}^* -simple germs are also those given in Table 2.*

The action \mathcal{K}^* does not only preserves the number of branches in each semi-space $z > 0$ and $z < 0$ of the zero fibre of any map-germ $\mathbb{R}^3, 0 \rightarrow \mathbb{R}^2, 0$, but also the contact of the zero fibre with the horizontal planes.

2.4 \mathcal{B} -singularities of the discriminant

We also can study the singularities of the discriminant up to an action that preserves the leaf of $\eta = dz$ through the origin considering the subgroup \mathcal{B} of \mathcal{A} which consists of pairs of diffeomorphisms (h, k) , where h preserves the plane $z = 0$ in the source and k is any diffeomorphism in the target. The group \mathcal{B} acts on $m(x, y, z). \mathcal{E}(3, 2)$ and is a Damon geometric subgroup. Theorem 2.3 models codimension less than or equal to four simple singularities of the discriminant map-germ up to subgroup \mathcal{B} .

Theorem 2.3 (Theorem 1.1, [3]) *The \mathcal{B} -simple map-germs $(\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0)$ of corank at most 1 and \mathcal{B}_e -codimension ≤ 4 are given in Table 3.*

Table 3: Normal forms of \mathcal{B}_e -codimension ≤ 4 ($\epsilon, \epsilon_1 = \pm 1$).

| Normal form | \mathcal{B}_e -codimension |
|--|------------------------------|
| (x, y) | 0 |
| $(x, z + \epsilon y^2)$ | 0 |
| $(x, z + xy + y^3)$ | 0 |
| $(x, z + y^3 + \epsilon^{k-1} x^k y), k \geq 2$ | $k - 1$ |
| $(x, z + xy + \epsilon y^4)$ | 1 |
| $(x, z + xy + y^5 + \epsilon y^7)$ | 2 |
| $(x, z + xy + y^5)$ | 3 |
| $(x, z + xy + \epsilon y^6 + y^9)$ | 4 |
| $(x, z + xy^2 + \epsilon y^4 + y^{2k+1}), k \geq 2$ | k |
| $(x, z + xy^2 + y^5 + \epsilon y^6)$ | 3 |
| $(x, z + xy^2 + y^5 + \epsilon y^9)$ | 4 |
| $(x, z + x^2 y + \epsilon y^4 + \epsilon_1 y^5)$ | 3 |
| $(x, z + x^2 y + \epsilon y^4)$ | 4 |
| $(x, y^2 + \epsilon z^2 + \epsilon_1^k x^{k-1} z), k \geq 2$ | $k - 2$ |
| $(x, y^2 + xz + \epsilon z^3)$ | 1 |
| $(x, y^2 + xz + \epsilon z^4 + \epsilon_1 z^6)$ | 2 |
| $(x, y^2 + xz + \epsilon z^4)$ | 3 |
| $(x, yz + xy + \epsilon y^3)$ | 1 |
| $(x, yz + xy + y^4 + \epsilon y^6)$ | 2 |
| $(x, yz + xy + y^4)$ | 3 |
| $(x, xy + z^2 + y^3 + \epsilon y^k z), k \geq 2$ | k |

We observe that the classification under the action of the group \mathcal{B} also models the singularities of projections of hypersurfaces with boundary in \mathbb{R}^4 to planes and those of invariant map-germs $\mathbb{R}^3 \rightarrow \mathbb{R}^2$. See [3] for details.

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